Given an input clock rate $r_{\texttt{in}} > 0$ and a divisor d define a frequency

$$r(d) := \frac{r_{\texttt{in}}}{2 \cdot d}$$

.

Now given a target frequency $r_{\texttt{target}} > 0$ find an integer divisor $d_{\texttt{int}}$ such that

$$e(d_{\texttt{int}}) := \left| \frac{1}{r_{\texttt{target}}} - \frac{1}{r(d_{\texttt{int}})} \right| = \left| \frac{1}{r_{\texttt{target}}} - \frac{2 \cdot d_{\texttt{int}}}{r_{\texttt{in}}} \right|$$

is minimal. The optimal divisor $d_{\texttt{opt}}$ is $\frac{r_{\texttt{in}}}{2 \cdot r_{\texttt{target}}}$ which might not be integer though.

The obvious candidates for $d_{\texttt{int}}$ are $d_\uparrow := \lceil d_{\texttt{opt}} \rceil$ and $d_\downarrow := \lfloor d_{\texttt{opt}} \rfloor$.

Now assuming $d \geq d_\uparrow$, we have:

$$\begin{aligned}
\frac{2 \cdot d}{r_{\texttt{in}}} &\geq \frac{2 \cdot d_\uparrow}{r_{\texttt{in}}} \\
&= \frac{2 \cdot \lceil d_{\texttt{opt}} \rceil}{r_{\texttt{in}}} \\
&\geq \frac{2 \cdot d_{\texttt{opt}}}{r_{\texttt{in}}} \\
&= \frac{2 \cdot \frac{r_{\texttt{in}}}{2 \cdot r_{\texttt{target}}}}{r_{\texttt{in}}} \\
&= \frac{1}{r_{\texttt{target}}} \\
\implies e(d) &= \left| \frac{1}{r_{\texttt{target}}} - \frac{2 \cdot d}{r_{\texttt{in}}} \right| \\
&= \frac{2 \cdot d}{r_{\texttt{in}}} - \frac{1}{r_{\texttt{target}}}
\end{aligned}$$

With this it's trivial to prove that $e(d_\uparrow) \leq e(d)$. An analogous calculation for $d \leq d_\downarrow$ shows $e(d_\downarrow) \leq e(d)$. So $d_\uparrow$ and $d_\downarrow$ are indeed the best integer approximations right and left of $d_{\texttt{opt}}$.

Now assume $d_\uparrow$ is a better approximation than $d_\downarrow$:

$$\begin{aligned}
e(d_\uparrow) &\leq e(d_\downarrow) \\
\iff \left| \frac{1}{r_{\texttt{target}}} - \frac{2 \cdot d_\uparrow}{r_{\texttt{in}}} \right| &\leq \left| \frac{1}{r_{\texttt{target}}} - \frac{2 \cdot d_\downarrow}{r_{\texttt{in}}} \right| \\
\iff \left| \frac{r_{\texttt{in}}}{2 \cdot r_{\texttt{target}}} - d_\uparrow \right| &\leq \left| \frac{r_{\texttt{in}}}{2 \cdot r_{\texttt{target}}} - d_\downarrow \right| \\
\iff \left| d_{\texttt{opt}} - d_\uparrow \right| &\leq \left| d_{\texttt{opt}} - d_\downarrow \right|
\end{aligned}$$

So $d_{\texttt{int}}$ is $d_{\texttt{opt}}$ rounded to the nearest integer.